The fixed point alternative and the stability of ternary derivations

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Abstract. Using fixed point methods, we prove the generalized Hyers–Ulam–Rassias stability of ternary derivations in ternary Banach algebras for the generalized Jensen–type functional equation

$$\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) = f(\mu x) \ .$$

1. Introduction

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A. Cayley [1] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([2]).
The comments on physical applications of ternary structures can be found in [3–12].
Let A be a Banach ternary algebra and X be a Banach space. Then X is called a ternary Banach A-module, if module operations $A\times A\times X \to X$, $A\times X\times A \to X$, and $X\times A\times A \to X$ which are $\mathbb{C}$-linear in every variable. Moreover satisfy

$$[[xab]_X\ cd]_X = [x[abc]_A\ d]_X = [xa[bcd]_A]_X,$$
$$[[axb]_X\ cd]_X = [a[xbc]_X\ d]_X = [ax[bcd]_A]_X,$$
$$[[abx]_X\ cd]_X = [a[bxc]_X\ d]_X = [ab[xcd]_X]_X,$$
$$[abc]_A\ xd]_X = [a[bcx]_X\ d]_X = [ab[cxd]_X]_X,$$
$$[[abc]_A\ dx]_X = [a[bcd]_A\ x]_X = [ab[cdx]_X]_X$$

for all $x \in X$ and all $a, b, c, d \in A$,

$$\max\{\|xab\|, \|axb\|, \|abx\|\} \leq \|a\|\|b\|\|x\|$$

for all $x \in X$ and all $a, b \in A$.
Let $(A, [\ ]_A)$ be a Banach ternary algebra over a scalar field $\mathbb{R}$ or $\mathbb{C}$ and $(X, [\ ]_X)$ be a ternary Banach A-module. A linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a ternary derivation, if

$$D([xyz]_A) = [D(x)yz]_X + [xD(y)z]_X + [xyD(z)]_X$$

for all $x, y, z \in A$.
A linear mapping $D : (A, [\ ]_A) \to (X, [\ ]_X)$ is called a ternary Jordan derivation, if

$$D([xxx]_A) = [D(x)xx]_X + [xD(x)x]_X + [xxD(x)]_X$$

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for all $x \in A$.
The stability of functional equations was first introduced by S. M. Ulam [13] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D. H. Hyers [14] gave a partial solution of $Ulam's$ problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1950, T. Aoki [25] was the second author to treat this problem for additive mappings (see also [16]). In 1978, Th. M. Rassias [17] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th. M. Rassias [17] is called the Hyers–Ulam–Rassias stability. According to Th. M. Rassias theorem:
Theorem 1.1. *Let $f : E \longrightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. If $p < 0$ then inequality (1.3) *holds for all $x, y \neq 0$, and* (1.4) *for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is linear.*

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [18–29].
C. Park [30] has contributed works to the stability problem of ternary homomorphisms and ternary derivations (see also [31]).

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (see also [32–38]).
In this paper, we will adopt the fixed point alternative of Cădariu and Radu to prove the generalized Hyers–Ulam–Rassias stability of ternary derivations on ternary Banach algebras associated with the following functional equation

$$\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) = f(\mu x) \ .$$

Throughout this paper, assume that $(A, [\ ]_A)$ is a ternary Banach algebra and X is a ternary Banach $A-$module.

2. Main Results

Before proceeding to the main results, we will state the following theorem.

Theorem 2.1. *(the alternative of fixed point [32]). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*
$d(T^m x, T^{m+1}x) = \infty$ *for all $m \geq 0$,*
or other exists a natural number m_0 such that

⋆ $d(T^m x, T^{m+1}x) < \infty$ *for all* $m \geq m_0$;
⋆ *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T;
⋆ y^* *is the unique fixed point of* T *in the set* $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;
⋆ $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

We start our work with the following theorem which establishes the generalized Hyers–Ulam–Rassias stability of ternary derivations.

Theorem 2.2. *Let* $f : A \to X$ *be a mapping for which there exists a function* $\phi : A^6 \to [0, \infty)$ *such that*

$$\begin{aligned} &\|\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) - f(\mu x) \\ &+ f([abc]_A) - [f(a)bc]_X - [af(b)c]_X - [abf(c)]_X\|_X \leq \phi(x,y,z,a,b,c), \end{aligned} \tag{2.1}$$

for all $\mu \in \mathbb{T}$ *and all* $x, y, z, a, b, c \in A$. *If there exists an* $L < 1$ *such that*

$$\phi(x,y,z,a,b,c) \leq 3L\phi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}, \frac{a}{3}, \frac{a}{3}, \frac{c}{3})$$

for all $x, y, z, a, b, c \in A$, *then there exists a unique ternary derivation* $D : A \to X$ *such that*

$$\|f(x) - D(x)\|_B \leq \frac{L}{1-L}\phi(x, 0, 0, 0, 0, 0) \tag{2.2}$$

for all $x \in A$.

Proof. It follows from

$$\phi(x,y,z,a,b,c) \leq 3L\phi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}, \frac{a}{3}, \frac{b}{3}, \frac{c}{3})$$

that

$$lim_j 3^{-j}\phi(3^j x, 3^j y, 3^i z, 3^i a, 3^i b, 3^i c) = 0 \tag{2.3}$$

for all $x, y, z, a, b, c \in A$.
Put $\mu = 1, y = z = a = b = c = 0$ in (2.1) to obtain

$$\|3f(\frac{x}{3}) - f(x)\|_B \leq \phi(x, 0, 0, 0, 0, 0) \tag{2.4}$$

for all $x \in A$. Hence,

$$\|\frac{1}{3}f(3x) - f(x)\|_B \leq \frac{1}{3}\phi(3x, 0, 0, 0, 0, 0) \leq L\phi(x, 0, 0, 0, 0, 0) \tag{2.5}$$

for all $x \in A$.
Consider the set $X' := \{g \mid g : A \to B\}$ and introduce the generalized metric on X':

$$d(h, g) := inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\|_B \leq C\phi(x, 0, 0, 0, 0, 0)\forall x \in A\}.$$

It is easy to show that (X', d) is complete. Now we define the linear mapping $J : X' \to X'$ by

$$J(h)(x) = \frac{1}{3}h(3x)$$

for all $x \in A$. By Theorem 3.1 of [32],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all $g, h \in X'$.
It follows from (2.5) that

$$d(f, J(f)) \leq L.$$

By Theorem 1.2, J has a unique fixed point in the set $X_1 := \{h \in X' : d(f,h) < \infty\}$. Let D be the fixed point of J. D is the unique mapping with

$$D(3x) = 3D(x)$$

for all $x \in A$ satisfying there exists $C \in (0,\infty)$ such that

$$\|D(x) - f(x)\|_B \leq C\phi(x,0,0,0,0,0)$$

for all $x \in A$. On the other hand we have $lim_n d(J^n(f), D) = 0$. It follows that

$$lim_n \frac{1}{3^n} f(3^n x) = D(x) \tag{2.6}$$

for all $x \in A$. It follows from $d(f,D) \leq \frac{1}{1-L} d(f, J(f))$, that

$$d(f,D) \leq \frac{L}{1-L}.$$

This implies the inequality (2.2). It follows from (2.1), (2.3) and (2.6) that

$$\begin{aligned}
&\|D(\frac{x+y+z}{3}) + D(\frac{x-2y+z}{3}) + D(\frac{x+y-2z}{3}) - D(x)\|_X \\
&= lim_n \frac{1}{3^n}\|f(3^{n-1}(x+y+z)) + f(3^{n-1}(x-2y+z)) + f(3^{n-1}(x+y-2z)) - f(3^n x)\|_X \\
&\leq lim_n \frac{1}{3^n}\phi(3^n x, 3^n y, 3^n z, 3^n a, 3^n b, 3^n c) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D(\frac{x+y+z}{3}) + D(\frac{x-2y+z}{3}) + D(\frac{x+y-2z}{3}) = D(x)$$

for all $x, y, z \in A$. Put $w = \frac{x+y+z}{3}, t = \frac{x-2y+z}{3}$ and $s = \frac{x+y-2z}{3}$ in above equation, we get $D(w+t+s) = D(w) + D(t) + D(s)$ for all $w, t, s \in A$. Hence, D is Cauchy additive. By putting $y = z = x, a = b = c = 0$ in (2.1), we have

$$\|\mu f(x) - f(\mu x)\|_X \leq \phi(x,x,x,0,0,0)$$

for all $x \in A$. It follows that

$$\|D(\mu x) - \mu D(x)\|_X = lim_n \frac{1}{3^n}\|f(\mu 3^n x) - \mu f(3^n x)\|_X \leq lim_n \frac{1}{3^n}\phi(3^n x, 3^n x, 3^n x, 3^n a, 3^n b, 3^n c) = 0$$

for all $\mu \in \mathbb{T}$, and all $x \in A$. One can show that the mapping $D : A \to B$ is $\mathbb{C}-$linear. It follows from (2.1) that

$$\begin{aligned}
&\|D([xyz]_A) - [D(x)yz]_X - [xD(y)z]_X - [xyD(z)]_X\|_X \\
&= lim_n \|\frac{1}{27^n} D([3^n x 3^n y 3^n z]_A) - \frac{1}{27^n}([D(3^n x)3^n y 3^n z]_X \\
&+ [3^n x D(3^n y) 3^n z]_X + [3^n x 3^n y D(3^n z)]_X)\|_X \leq lim_n \frac{1}{27^n}\phi(0,0,0,3^n x, 3^n y, 3^n z) \\
&\leq lim_n \frac{1}{3^n}\phi(0,0,0,3^n x, 3^n y, 3^n z) \\
&= 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D([xyz]_A) = [D(x)yz]_X + [xD(y)z]_X + [xyD(z)]_X$$

for all $x, y, z \in A$. Hence, $D : A \to X$ is a ternary derivation satisfying (2.2), as desired.

$\square$

We prove the following Hyers–Ulam–Rassias stability problem for ternary derivations on ternary Banach algebras.

Corollary 2.3. *Let $p \in (0,1), \theta \in [0,\infty)$ be real numbers. Suppose $f : A \to X$ satisfies*

$$\|\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) - f(\mu x)\|_X \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$

$$\|f([abc]_A) - [f(a)bc]_X - [af(b)c]_X - [abf(c)]_X\|_X, \leq \theta(\|a\|_A^p + \|b\|_A^p + \|c\|_A^p),$$

for all $\mu \in \mathbb{T}$ and all $a,b,c,x,y,z \in A$. Then there exists a unique ternary derivation $D : A \to X$ such that

$$\|f(x) - D(x)\|_B \leq \frac{2^p\theta}{2-2^p}\|x\|_A^p$$

for all $x \in A$.

Proof. Setting $\phi(x,y,z,a,b,c) := \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|a\|_A^p + \|b\|_A^p + \|c\|_A^p)$ all $x,y,z,a,b,c \in A$. Then by $L = 2^{p-1}$, we get the desired result. □

Theorem 2.4. *Let $f : A \to X$ be a mapping for which there exists a function $\phi : A^6 \to [0,\infty)$ satisfying (2.1). If there exists an $L < 1$ such that $\phi(x,y,z,a,b,c) \leq \frac{1}{3}L\phi(3x,3y,3z,3a,3b,3c)$ for all $x,y,z,a,b,c \in A$, then there exists a unique ternary derivation $D : A \to X$ such that*

$$\|f(x) - D(x)\|_X \leq \frac{L}{3-3L}\phi(x,0,0,0,0,0) \tag{2.7}$$

for all $x \in A$.

Proof. It follows from (2.4) that

$$\|3f(\frac{x}{3}) - f(x)\|_X \leq \phi(\frac{x}{3},0,0,0,0,0) \leq \frac{L}{3}\phi(x,0,0,0,0,0) \tag{2.8}$$

for all $x \in A$. We consider the linear mapping $J : X' \to X'$ such that

$$J(h)(x) = 3h(\frac{x}{3})$$

for all $x \in A$. It follows from (2.9) that

$$d(f, J(f)) \leq \frac{L}{3}.$$

By Theorem 2.1, J has a unique fixed point in the set $X_1 := \{h \in X' : d(f,h) < \infty\}$. Let D be the fixed point of J, that is,

$$D(3x) = 3D(x)$$

for all $x \in A$ satisfying there exists $C \in (0,\infty)$ such that

$$\|D(x) - f(x)\|_X \leq C\phi(x,0,0,0,0,0)$$

for all $x \in A$. We have $d(J^n(f), D) \to 0$ as $n \to 0$. This implies the equality

$$lim_n 3^n f(\frac{x}{3^n}) = D(x) \tag{2.9}$$

for all $x \in A$. It follows from $d(f,D) \leq \frac{1}{1-L}d(f,J(f))$, that

$$d(f,D) \leq \frac{L}{3-3L},$$

which implies the inequality (2.7). The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $p \in (3,\infty), \theta \in [0,\infty)$ be real numbers. Suppose $f : A \to X$ satisfies*

$$\|\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) - f(\mu x)\|_X \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$
$$\|f([abc]_A) - [f(a)bc]_X - [af(b)c]_X - [abf(c)]_X\|_X, \leq \theta(\|a\|_A^p + \|b\|_A^p + \|c\|_A^p),$$

for all $\mu \in \mathbb{T}$ and all $a, b, c, x, y, z \in A$.

Then there exists a unique ternary derivation $D : A \to X$ such that

$$\|f(x) - D(x)\|_X \leq \frac{\theta}{3^p - 3}\|x\|_A^p$$

for all $x \in A$.

Proof. Setting $\phi(x,y,z,a,b,c) := \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|a\|_A^p + \|b\|_A^p + \|c\|_A^p)$ for all $x, y, z, a, b, c \in A$. Then by $L = 3^{1-p}$, we get the desired result. □

Now we investigate the generalized Hyers–Ulam–Rassias stability of Jordan ternary derivations.

Theorem 2.6. *Let $f : A \to X$ be a mapping for which there exists a function $\phi : A^4 \to [0,\infty)$ such that*

$$\begin{aligned}&\|\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) - f(\mu x)\\ &+f([aaa]_A) - [f(a)aa]_X - [af(a)a]_X - [aaf(a)]_X\|_X \leq \phi(x,y,z,a),\end{aligned} \tag{2.10}$$

for all $\mu \in \mathbb{T}$ and all $x, y, z, a, b, c \in A$. If there exists an $L < 1$ such that

$$\phi(x,y,z,a) \leq 3L\phi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}, \frac{a}{3})$$

for all $x, y, z, a \in A$, then there exists a unique Jordan ternary derivation $D : A \to X$ such that

$$\|f(x) - D(x)\|_X \leq \frac{L}{1-L}\phi(x,0,0,0) \tag{2.11}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.2, there exists a unique involutive $\mathbb{C}-$linear mapping $D : A \to X$ satisfying (2.11). The mapping D is given by

$$D(x) = lim_n \frac{1}{3^n} f(3^n x)$$

for all $x \in A$. The relation (2.10) follows that

$$\begin{aligned}&\|D([xxx]_A) - [D(x)xx]_X - [xD(x)x]_X - [xxD(x)]_X\|_X\\ &= lim_n \|\frac{1}{27^n} D([3^n x 3^n x 3^n x]_A) - \frac{1}{27^n}([D(3^n x)3^n x 3^n x]_X\\ &+ [3^n x D(3^n x) 3^n x]_X + [3^n x 3^n x D(3^n x)]_X)\|_X \leq lim_n \frac{1}{27^n}\phi(0,0,0,3^n x)\\ &\leq lim_n \frac{1}{3^n}\phi(0,0,0,3^n x)\\ &= 0\end{aligned}$$

for all $x \in A$. So

$$D([xxx]_A) = [D(x)xx]_X + [xD(x)x]_X + [xxD(x)]_X$$

for all $x \in A$. Hence, $D : A \to X$ is a Jordan ternary derivation satisfying (2.11), as desired. $\square$

We prove the following Hyers–Ulam–Rassias stability problem for Jordan ternary derivations on ternary Banach algebras.

Corollary 2.7. *Let $p \in (0,1), \theta \in [0,\infty)$ be real numbers. Suppose $f : A \to B$ satisfies*

$$\|\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) - f(\mu x)\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$

$$\|f([xxx]_A) - [f(x)xx]_X - [xf(x)x]_X - [xxf(x)]_X\|_X \leq 3\theta(\|x\|_A^p)$$

for all $\mu \in \mathbb{T}$, and all $x, y, z \in A$. Then there exists a unique Jordan ternary derivation $D : A \to X$ such that

$$\|f(x) - D(x)\|_X \leq \frac{2^p\theta}{2-2^p}\|x\|_A^p$$

for all $x \in A$.

Proof. Setting $\phi(x,y,z,a) := \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|a\|_A^p)$ all $x, y, z, a \in A$. Then by $L = 2^{p-1}$, we get the desired result. $\square$

Theorem 2.8. *Let $f : A \to X$ be a mapping for which there exists a function $\phi : A^4 \to [0,\infty)$ satisfying (2.10). If there exists an $L < 1$ such that $\phi(x,y,z,a) \leq \frac{1}{3}L\phi(3x,3y,3z,3a)$ for all $x, y, z, a \in A$, then there exists a unique Jordan ternary derivation $D : A \to X$ such that*

$$\|f(x) - D(x)\|_X \leq \frac{L}{3-3L}\phi(x,0,0,0)$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 2.6. $\square$

Corollary 2.9. *Let $p \in (3,\infty), \theta \in [0,\infty)$ be real numbers. Suppose $f : A \to X$ satisfies*

$$\|\mu f(\frac{x+y+z}{3}) + \mu f(\frac{x-2y+z}{3}) + \mu f(\frac{x+y-2z}{3}) - f(\mu x)\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p),$$

$$\|f([xxx]_A) - [f(x)xx]_X - [xf(x)x]_X - [xxf(x)]_X\|_X \leq 3\theta(\|x\|_A^p)$$

for all $\mu \in \mathbb{T}$, and all $x, y, z \in A$. Then there exists a unique Jordan ternary derivation $D : A \to X$ such that

$$\|f(x) - D(x)\|_X \leq \frac{\theta}{3^p-3}\|x\|_A^p$$

for all $x \in A$.

Proof. Setting $\phi(x,y,z,a) := \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|a\|_A^p)$ all $x, y, z \in A$ in above theorem. Then by $L = 3^{1-p}$, we get the desired result. $\square$

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